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Predictability in spatially extended systems

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Abstract. The predictability problem is discussed in turbulent fluids and in spatially extended systems. We perform a numerical analysis of the spatial propagation of a small perturbation initially localized at a point of a one-dimensional lattice of coupled maps, where the interactions can be either local or non-local in space. In both cases, the nonlinear terms in the dynamics of the perturbation cannot be neglected and the predictability problem cannot be reduced to the linearized evolution equations. Only for non-local interactions the predictability time is proportional to the inverse Lyapunov exponent λ, as in dynamical systems with few degrees of freedom. For local interactions it is proportional to the size of the system and is practically independent of the chaoticity degree λ.

The possibility of making predictions on the future state of a dynamical system has severe limitations which stems from the presence of deterministic chaos, in the sense of sensitive dependence on initial conditions (Lorenz 1963). The common belief is that the predictability time should be proportional to the inverse of the typical rate of divergence of nearby trajectories, as a consequence of the following considerations. Let us consider a dynamical system given by a map

$$x_i(t+1) = g_i(\boldsymbol{x}(t)) \qquad i = 1, \cdots, N \tag{1}$$

where the time t is discrete and $\boldsymbol{x} = (x_1, \cdots x_N)$. As long as a perturbation $\delta\boldsymbol{x}$ is small enough, its behaviour is ruled by the linearized evolution equations for the tangent vector $\boldsymbol{z}$

$$z_i(t+1) = \sum_i^N \left.\frac{\mathrm{d}g_i}{\mathrm{d}x_j}\right|_{\boldsymbol{x}(t)} z_j(t). \tag{2}$$

In a similar way, one can consider the flow given by the differential equations $\mathrm{d}x_i/\mathrm{d}t = G_i(\boldsymbol{x})$ and $\mathrm{d}z_i/\mathrm{d}t = \sum_i^N \mathrm{d}G_i/\mathrm{d}x_j|_{\boldsymbol{x}(t)}\, z_j(t)$.

In chaotic systems, one has $|\boldsymbol{z}(t)| \sim \exp(\lambda\, t)$ (where λ is the maximum Lyapunov exponent) for almost all initial conditions, when $t \to \infty$. It follows that, if the maximum admitted error on the knowledge of the state of the system is δ_{max} and the initial error is δ_0, the future can be predicted up to a time

$$T_{\mathrm{p}} \sim \frac{1}{\lambda} \ln \frac{\delta_{\mathrm{max}}}{\delta_0}. \tag{3}$$

This simple remark is important since it implies that in dynamical systems the forecasting is mainly limited by the chaotic nature of the evolution and very weakly by the resolution of the measurements. The gain obtained by achieving finer resolutions is only logarithmic

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and can be safely ignored for practical purposes. The problem of predictability in low-dimensional systems is thus solved in the context of the theory of chaotic dynamics by considering the evolution of an *infinitesimal* perturbation on the original flow.

However, it is not obvious that the predictability time can be expressed in terms of the Lyapunov exponent in partial differential equations or in dynamical systems with a large number of degrees of freedom, where one should consider not only the rate of divergence of nearby trajectories, but also the direction in the phase space where a perturbation grows. In particular, for spatially extended systems the propagation in space of a perturbation might be unrelated to the chaoticity degree of the global dynamical system.

This point can be easily understood and analysed in the context of coupled maps on a lattice. If the interactions among the maps are local, the propagation of a perturbation is not directly related to the Lyapunov exponent. One expects that it propagates from one lattice point toward the boundaries at the 'sound' speed. It is important to stress that this is not the case for differential equations, where one cannot exclude a spreading of a perturbation in an infinitesimal time, even for local interactions.

On the other hand, we shall see that for non-local interactions the situation is even less clear and should be discussed case by case.

Let us briefly review the different reasons for a failure of the relation (3) between predictability and maximum Lyapunov exponent in a generic dynamical system:

(a) The Lyapunov exponent λ is a global quantity: it measures the average exponential rate of divergence of nearby trajectories. In general there are finite-time fluctuations of this rate and it is possible to define an 'instantaneous' rate, called effective Lyapunov exponent (Paladin and Vulpiani 1987)

$$\gamma_{x(t)} = \frac{1}{\tau} \ln \frac{|z(t+\tau)|}{|z(t)|} \tag{4}$$

which depends (for finite delay time τ) on the particular point of the trajectory $x(t)$ where the perturbation is performed. In the same way, the predictability time T_p fluctuates, following the γ-variations.

(b) In dynamical systems with many degrees of freedom, the interactions among different parts of the system play an important role in the growth of the perturbation. The knowledge of the statistics of the effective Lyapunov exponent is insufficient and one has to analyse the behaviour of the tangent vector $z(t)$, which gives the direction along which an infinitesimal perturbation grows (see e.g. Pikovsky 1993). Moreover, one is often interested in the case of a perturbation concentrated on certain degrees of freedom (e.g. small-length scales in weather forecasting), and of a prediction on the evolution of other degrees of freedom (e.g. large-length scales). In the framework of a linear approximation for the evolution of δx, the relevant quantity is the time T_R, necessary for the tangent vector to relax on the time-dependent eigenvector $e(t)$ of the stability matrix, corresponding to the maximum Lyapunov exponent. It is worth stressing that a generic tangent vector $z(t)$ relaxes exponentially fast to $e(t)$ (Orszag *et al* 1987). When the perturbations on a system are small enough, and a linear approach can be used, nonlinear terms of type $\delta x_i\, \delta x_j$ are negligible in the evolution equation of δx, and one has

$$T_p \sim T_R + \frac{1}{\lambda} \ln \frac{\delta_{max}}{\delta_0}. \tag{5}$$

In conclusion, the mechanism of transfer of the error δx through the degrees of freedom of the system could be more important than the rate of divergence of nearby trajectories.

(c) The nonlinear terms in the evolution equations of the perturbation δx may have a key role in the predictability problem, so that the behaviour of δx is not determined by the evolution of the tangent vector z.

In one of the first attempts to formulate a theory of the predictability in turbulent flows, Lorenz takes into account (b) and (c), on physical grounds. It is instructive to briefly discuss his phenomenological ideas (Lorenz 1969, Leith and Kraichnan 1972, Lilly 1979). In the energy cascade for three-dimensional turbulence it is assumed that the time $\tau(k)$ necessary for a perturbation at wavenumber $2k$ in order to induce a complete uncertainty on the velocity field on scales corresponding to wavenumber k, is proportional to the eddy turn-over time at scale k. By using dimensional arguments, one gets

$$\tau(k) = \frac{1}{k\, v_k}$$

where v_k is the typical velocity difference on length scale k^{-1}. The smallest characteristic scale of the system is the Kolmogorov length $\eta \sim {k_K}^{-1}$, where viscosity overwhelms the nonlinear transfer of energy. As a consequence, the predictability time, understood as the time necessary for an incertitude on the Kolmogorov length to propagate up to the largest characteristic length scale $L_0 \sim {k_0}^{-1}$ (the scale of the energy containing eddies) is

$$T_P = \sum_{n=0}^{N} \tau(2^n\, k_K) \tag{6}$$

where $N = \ln(k_K/k_0) = \ln(L_0/\eta)$. In the Kolmogorov theory K41, the eddy turn-over time follows the law $\tau(k) \sim k^{-2/3}$ and the Kolmogorov length vanishes as a power of the Reynolds number Re, $\eta \sim Re^{-3/4}$. Lorenz thus postulates the existence of an inverse cascade of the error in $N \sim \ln Re$ steps. The predictability time obtained by (6) is proportional to the turn-over time of the energy containing eddies,

$$T_{\mathrm{p}} \sim L_0 \,/\, v_0 \tag{7}$$

and therefore is independent of Reynolds. This happens because the characteristic life-times $\tau(\ell \sim k^{-1}$, associated with the eddy structures of lengths $\ell \ll L_0$ are negligible with respect to the 'macroscopic' time scale L_0/v_0.

Relation (7) is questionable, however, since the Lorenz approach involves many characteristic times and assumes a precise physical mechanism for the inverse cascade. In fact, a numerical analysis of a simplified dynamical system (the GOY shell model) (Gledzer 1973, Ohkitani and Yamamada 1987, 1988) for the turbulence cascade, shows that the predictability time of the model is proportional to the inverse of the maximum Lyapunov exponent (Crisanti *et al* 1993a, b). In turbulence, one expects that λ^{-1} is given by the smallest characteristic time $\tau(k_K)$, so that in the K41 theory $\lambda \sim Re^{1/2}$ (Ruelle 1979) while in the multifractal theory one has the prediction $\lambda \sim Re^{0.46}$, where the slight correction to the power is due to the intermittency of the energy dissipation (Crisanti *et al* 1993a, b). In the GOY model, which seems to be multifractal (Jensen *et al* 1991) we have numerically verified that

$$T_{\mathrm{p}} \sim \lambda^{-1} \sim Re^{-0.46}. \tag{8}$$

The Lorenz theory and the results obtained in the shell model lead to very different pictures of predictability in turbulence and it is quite difficult to decide which is the correct one. Let us remark that both (7), where $T_p \sim L_0/v_0 \gg \lambda^{-1}$, and (8) where $T_p \sim \lambda^{-1}$, are limiting cases of relation (5). It is worth stressing that the assumption of locality of the transfer mechanism of the perturbation in k-space is at the basis of (7). Non-local mechanism, which could be present in real fluids, would lead to predictability of the type of the shell model, where the chaotic nature of the evolution equation has the most relevant effect in determining the predictability properties.

The same kind of problem arises when we consider real space instead of the Fourier k-space. We have thus decided to study a system of coupled maps on a one-dimensional lattice which is the simplest model which exhibits the two different predictability scenarios.

The equations of our toy model are

$$x_i(t+1) = (1-\epsilon_0)\, f(x_i(t)) + \frac{1}{2}\sum_{j=1}^{N/2-2} \epsilon_j \, \{f(x_{i+j}(t)) + f(x_{i-j}(t))\} \tag{9}$$

where f is a chaotic map of the interval [0, 1] into itself, $\epsilon_0 = \sum_{j=1}^{N/2-2} \epsilon_j$, and periodic boundary conditions $x_i = x_{i\pm N}$ are assumed.

We consider the case of nearest-neighbour interactions ($\epsilon_j = 0$ if $j \geqslant 2$) and of couplings of the type

$$\epsilon_1 = C_1 \qquad \text{and} \qquad \epsilon_j = \frac{C_2}{j^\alpha} \qquad \text{for } \; j \geqslant 2 \tag{10}$$

where the power α measures the strength of non-locality.

There is no strong relation between the the evolution laws (9) and those of a fluid. However, the system of coupled maps can be regarded as a discrete (time-space) version of an integro-differential equation. Let us recall that the Navier–Stokes equations for an incompressible fluid can be written as an integro-differential equation with non-local interactions between the points of the fluid.

In particular, we show here numerical results for the predictability using the logistic map

$$f(x) = r\,x\,(1-x) \qquad \text{mod } 1 \qquad r = 4$$

in equations (9). The predictability in the coupled map system seems not to be sensitive to the choice for f. For instance, we have observed the same qualitative behaviour using the logistic map with different values of the control parameter r or piecewise-linear maps of the interval into itself such as the Bernoulli map $f = 2x$ mod 1 and asymmetric tent maps.

The Lyapunov exponent of the coupled map system (9) is practically independent of N for both local and non-local interactions. This independence has also been found in other systems of coupled maps with local interactions (Kaneko 1989, Livi *et al* 1987).

The perturbation at initial time is performed at the centre of the lattice $i = N/2$, that is

$$|\delta x_{N/2}(0)| = \delta_0 \qquad \delta x_i(0) = 0 \qquad \text{for } \; i \neq N/2. \tag{11}$$

We then look at the time T_p needed for the perturbation to reach a certain threshold δ_{max} on the boundary of the lattice, that is, the maximum time t such that $|\delta x_1(t)| \leqslant \delta_{max}$.

In terms of the energy cascade in turbulent fluids which motivated our model, we are looking at the 'butterfly effect' starting from the centre of the lattice (which corresponds to the small-length scales) and arriving up to the first site (which is the analogue of the large scale).

For a system of maps coupled via a nearest-neighbour interaction, it is trivial to conclude that $\delta x_1(t) = 0$ for times $t < N/2$. Indeed, by a numerical iteration of (9) one observes that $\delta x_1(t) = 0$ for times $t < t^* = CN$. More precisely, for any fixed $\delta_0 > 0$, one has $|\delta x_1(t)| < \delta_0$ for $t < t^*$ where $t^*/N \to C$ for $N \to \infty$. For longer times, the perturbation starts to grow as a consequence of the chaotic dynamics, $\delta x_1(t) \sim \delta_0 e^{\lambda(t-t^*)}$. For local interactions, the predictability therefore is mainly determined by the waiting time t^* which is roughly proportional to the system size N, as shown in figure 1, which shows the predictability time $\langle T_P \rangle$ averaged over a large number of initial conditions. It can be fitted by the linear law

$$\langle T_P \rangle = t_1 + CN \tag{12}$$

where the time $t_1 \sim \lambda^{-1}$ is due to the exponential error growth after the waiting time and can be neglected when the lattice is large enough. This result in real space is analogous to the Lorenz formula (7), where the Lyapunov exponent has no relevant role and the predictability is determined by the waiting time necessary for a perturbation to pass from the small- to the large-length scales through the inverse cascade mechanism.

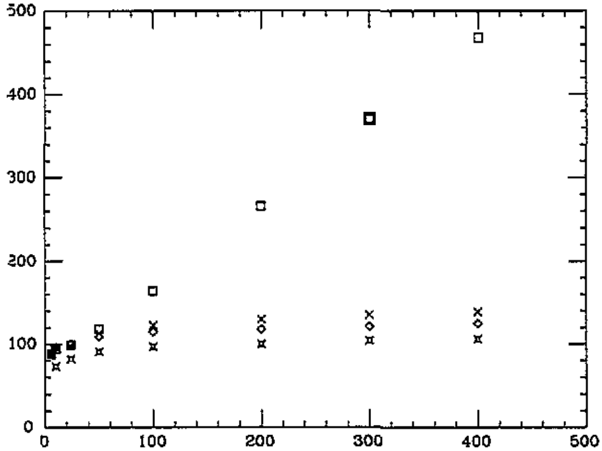

Figure 1. Average predictability time $\langle T_p \rangle$ versus N for: local coupling $C_1 = 0.3$ (squares); non-local coupling $C_1 = 0.3$, $C_2 = 0.01$ with $\alpha = 2$ (crosses) and $\alpha = 3$ (diamonds); mean field coupling $\epsilon_i = C_2/N$ and $C_2 = 0.3$ (crossed squares). The initial perturbation is performed at the centre of the lattice (site $i = N/2$) and has an amplitude 10^{-14}; the maximum admitted error is $\delta_{max} = 0.1$.

It is evident that the situation may be very different in the case of non-local interactions, since the perturbation on the centre of the lattice may propagate toward the boundaries without any time delay, due to the system size. The numerical integration of (9) shows that even for weak non-locality (e.g. $C_2 \ll C_1$ and rather large α-values), the waiting time t^* does not increase with the system size N and

$$\langle T_p \rangle \sim t_1 \sim \lambda^{-1}. \tag{13}$$

As shown in figure 1, the same qualitative behaviour is given by weakly non-local coupling and mean field interactions ($\epsilon_j = C_2/N$).

On the contrary, interactions with exponential decay or power-law decay with very large α give the same results as local interactions, suggesting that there is a threshold α_c

between local and non-local predictability. It would be interesting to understand what the dependence of α_c on C_1 and C_2 is, as well as the dependence on the shape of f. However, it is very difficult to decide this issue on a numerical basis.

In conclusion, the predictability time in spatially extended systems is given by two contributions: the waiting time t^* which is the maximum time for which $|\delta x(t)| < \delta_0$, and the characteristic time $t_1 \sim \lambda^{-1}$ associated with chaoticity degree. For non-local interactions, the waiting time is constant with respect to the size of the system N while for local interactions it is proportional to N. Let us stress that in these results the nonlinear terms in the evolution of a small perturbation $\delta x(t)$ are quite important. One numerically observes that the waiting time t^* is not just the relaxation time T_R of δx on the tangent eigenvector. In fact, we find that T_R is much larger than t^*.

Basically the mechanism of the perturbation growth is as follows. At the beginning the instability at the centre of the lattice δx_j with $j \approx N/2$ increases exponentially at a rate given by the maximum Lyapunov exponent. It thus attains the saturation level in a time $\sim \lambda^{-1}$. Only after the saturation at the lattice centre, the perturbation starts to spread (either in a slow or in fast way, according to the nature of the coupling). In a time t^* it reaches the boundary of the system.

If one assumes that the perturbation is infinitesimal and considers the linearized evolution of δx, i.e of the tangent vector, one gets a relaxation toward a typical stability eigenvector (corresponding to the maximum Lyapunov) whose components are of the order of $N^{-1/2}\exp(\lambda t)$ in a time $T_R \gg t^*$. The spatial spreading is thus accelerated by the nonlinear terms of the evolution law of δx and as a consequence, the predictability time decreases in a substantial way.

In physical phenomena, it is not clear whether interactions are local or non-local. In turbulence, for instance, one can think that non-local interactions among different points of the fluid arise as a consequence of the incompressibility condition, so that the problem of the relation between Lyapunov and predictability remains a crucial and controversial point. We have recently argued that the Lyapunov exponent is related to the predictability time on the basis of a numerical study of a shell model for the energy cascade in three-dimensional turbulence. As the Lyapunov exponent is found to be roughly proportional to the square root of the Reynolds number Re, we expect that the predictability time also decreases as $Re^{-1/2}$. On the other hand, Lorenz and other authors suggested that the predictability is substantially independent of Re, and is related to the lifetime of the largest eddies. Our results on coupled maps indicate that it is not easy to decide the correct picture for the growth of instabilities without a careful analysis of the nature of interactions in the Navier–Stokes equations and that one needs an experimental or numerical verification to discriminate between the two different scenarios.

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